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The Existence of Optimal Controls in the Absence of Convexity Conditions

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I. INTRODUCTION

We consider the existence of solutions of the following problem in the theory of optimal control. Let E_m denote the Euclidean m-space, and let there be given a compact interval $\Gamma = [t_1, t_2] \subset E_1$ (we do not exclude the case $t_1 = t_2$), a real number $t_0 \leq t_1$, a set $U \subset E_r$, a point $x^0 \in E_n$, a family $\mathscr{T}$ of closed sets $T_t \subset E_n$ defined for $t \in \Gamma$, a function $f(x, u, t)$ from $E_n \times U \times [t_0, t_2]$ to E_n, and a function $f_0(x, u, t)$ from $E_n \times U \times [t_0, t_2]$ to E_1. We assume that f_0 and f are continuous with continuous partial derivatives with respect to the coordinates x_i of x.

Measurable functions $u(t)$, defined for $t_0 \leq t \leq \bar{t}$ where $\bar{t} \in \Gamma$, whose range is contained in U will be called *admissible*. If $u(t)$ is admissible, we say that $x(t)$ is the trajectory corresponding to $u(t)$ if $x(t)$ is an n-dimensional, absolutely continuous vector function satisfying the relation $x(t_0) = x^0$ and the equation

$$\dot{x}(t) = f(x(t), u(t), t) \tag{1}$$

almost everywhere in $[t_0, \bar{t}]$, the interval on which $u(t)$ is defined. The absolutely continuous, scalar-valued function $x_0(t)$ which satisfies the relation $x_0(t_0) = 0$ and the equation

$$\dot{x}_0(t) = f_0(x(t), u(t), t) \tag{2}$$

almost everywhere in $[t_0, \bar{t}]$—where $x(t)$ is the trajectory corresponding to $u(t)$—will be termed the *cost function* corresponding to $u(t)$, and the number $x_0(\bar{t}) = C(u)$, the corresponding *cost*.

We say that an admissible control $u(t)$, defined for $t_0 \leq t \leq \bar{t} \in \Gamma$, transfers x^0 to $\mathscr{T}$ if the trajectory $x(t)$ corresponding to $u(t)$ satisfies the relation $x(\bar{t}) \in T_{\bar{t}}$. In this case, we shall also say that $u(t)$ transfers x^0 to the point $x(\bar{t})$ in the time $\bar{t}$. The optimal control problem consists in finding an admissible "control" function which transfers x^0 to $\mathscr{T}$ with minimal cost. Thus,

$u^*(t)$ is an "optimal control" if it is admissible, transfers x^0 to $\mathscr{T}$, and if $C(u^*) \leq C(u)$ for every admissible control $u(t)$ which transfers x^0 to $\mathscr{T}$.

The existence problem consists in answering the following two questions:

1. Do there exist any admissible "controls" $u(t)$ which transfer x^0 to $\mathscr{T}$?

2. Assuming at least one such control exists, does there exist an optimal control?

We shall confine ourselves to the second question.

Historically, the optimal control problem first arose as the "time-optimal" control problem, the problem in which $f_0 \equiv 1$ [so that $C(u) = \bar{t} - t_0$, the transfer time] and $t_0 = t_1$. Bellman, Glicksberg, and Gross [1] and Gamkrelidze [2] proved the existence of time-optimal controls if $f(x, u, t)$ has the form $f = Ax + Bu$ with A and B constant matrices, the set U is the cube $| u_i | \leq 1$ $(i = 1, \cdots, r)$, and T_t consists of a single point which is independent of t. These results were extended to the nonautonomous case—wherein A and B, as well as the point constituting the sets T_t, depend continuously on t—by Krasovskii [3] and LaSalle [4, 5]. Another extension, to the case where U is a general convex polyhedron having the origin as an interior point, was given by Pontryagin [6]. A very general existence theorem for time-optimal controls was proved by Filippov [7]. The only restrictions he imposed on f and U were that U be compact, that the sets

$$f(x, U, t) = \{f(x, u, t) : u \in U\}$$

be convex for every x and t, and that f not grow too rapidly with $\| x \|$. In addition, he assumed that the set of points

$$T = \{(t, x) : t \in [t_0, t_2] \text{ and } x \in T_t\}$$

be closed in E_{n+1}.

An existence theorem for the general (not necessarily time) optimal problem was first proved by Lee and Markus [8] under the assumptions that f and f_0 are linear in u [i.e., of the form $g(t, x) + h(t, x)\, u$], that U is compact and convex [so that the sets $f(x, U, t)$ are also convex], that the trajectories $x(t)$ corresponding to admissible controls are uniformly bounded, and that the sets T_t are bounded and vary continuously with t. Roxin [9] extended this result by replacing the conditions that f and f^0 be linear in u and that U be convex by the condition that the sets $\mathbf{f}(x, U, t)$ be convex, where $\mathbf{f} = (f_0, f)$. A similar result was derived by Warga [10] using a proof similar to the one of Filippov [7].

All of the existence theorems cited above are based on the convexity and compactness of the sets $\mathbf{f}(x, U, t)$. However, there is no reason to expect that the equations which arise from a physical problem will satisfy the convexity constraint. Warga [10] suggests that a system which does not satisfy the convexity property should be "relaxed" by enlarging the set of allowed values of $\dot{\mathbf{x}}(t) = (\dot{x}_0, \dot{x})$ from $\mathbf{f}(x(t), U, t)$ to the closure of the convex hull of $\mathbf{f}(x(t), U, t)$. He then shows that solutions of the relaxed problem can be uniformly approximated by solutions of the original problem. Gamkrelidze [11] suggests a method of modifying Eq. (1) and (2) to achieve the convexity condition, and then constructs solutions of the original problem which approximate solutions of the modified problem arbitrarily closely. He refers to the limit of these approximating solutions, in which the control must switch "infinitely often," as a "sliding regime."

Thus, in general, optimal controls only exist if the given problem is relaxed, or if sliding regimes are allowed. The principal result in this paper is that if the function $\mathbf{f}$ is linear in x, i.e., if

$$\dot{\mathbf{x}} = A(t)\,x + \boldsymbol{\varphi}(u, t), \tag{3}$$

then optimal controls exist if the sets $\boldsymbol{\varphi}(U, t)$ are only compact (and not necessarily convex). Indeed, for such systems nothing is gained by relaxing the problem or by introducing sliding regimes.

This result, whose proof is given in the next section, can be obtained by a slight extension of a theorem of Blackwell [12].[1]

II. Properties of the Attainable Set

We consider the case where Eq. (1) and (2) have the form

$$\begin{aligned} \dot{x} &= A(t)\,x(t) + \varphi(u, t) \\ \dot{x}_0 &= \alpha(t)\,x(t) + \varphi_0(u, t), \end{aligned} \tag{4}$$

where x, φ, and α are n-vectors, x_0 and φ_0 are scalars, and A is an $n \times n$ matrix. We assume that A, α, φ, and φ_0 are continuous functions. If we denote by $\mathbf{x}$ the $(n + 1)$-vector (x_0, x), Eq. (4) can be combined into the one equation

$$\dot{\mathbf{x}} = \mathbf{A}(t)\,\mathbf{x}(t) + \boldsymbol{\varphi}(u, t), \tag{5}$$

where $\mathbf{A}$ and $\boldsymbol{\varphi}$ are continuous.

We assume that the set U is compact. Hence, the sets $\boldsymbol{\varphi}(U, t)$ are also compact.

We shall use the concept of the attainable set as first introduced by Roxin [9]. We henceforth suppose that the initial time t_0 and initial point x^0 are fixed. Then, we say that the point $(t_1, \mathbf{x}^1) = (t_1, x_{01}, x^1) \in E_{n+2}$ is

[1] Blackwell's result was brought to my attention by H. Halkin, whose paper on a related topic [13] motivated much of the work in this paper.

attainable if there is an admissible control $u(t)$ which transfers x^0 to the point x^1 in the time t_1 with cost $C(u(t)) = x_{01}$. The *attainable set* $\mathscr{A}$ is then defined by:

$$\mathscr{A} = \{(t, \mathbf{x}) : t \in \Gamma, (t, \mathbf{x}) \text{ attainable}\}.$$

We define the fixed-time cross section $\mathscr{A}_\tau$ of $\mathscr{A}$, where $\tau \in \Gamma$, to be the set of all points $\mathbf{x} \in E_{n+1}$ for which $(\tau, \mathbf{x}) \in \mathscr{A}$.

THEOREM 1. *Under the above assumptions, the attainable set $\mathscr{A}$ is compact, and the fixed-time cross sections of $\mathscr{A}$ are convex and compact.*

PROOF. By the well-known variation of parameters formula,

$$\mathscr{A}_\tau = \left\{ X(\tau) \left[\mathbf{x}^0 + \int_{t_0}^{\tau} X^{-1}(t) \boldsymbol{\varphi}(u(t), t)\, dt \right] : u(t) \text{ admissible} \right\},$$

where $\mathbf{x}^0 = (0, x^0)$, and $X(t)$ is the matrix solution of the equation

$$\dot{X}(t) = \mathbf{A}(t)\, X(t), \qquad X(t_0) = I, \qquad \text{the identity matrix.}$$

To show that $\mathscr{A}_\tau$ is compact and convex, it is sufficient to prove that the set

$$\Lambda_\tau = \left\{ \int_{t_0}^{\tau} X^{-1}(t) \boldsymbol{\varphi}(u(t), t)\, dt : u(t) \text{ admissible} \right\}$$

is compact and convex. First, consider the set

$$R_\tau = \left\{ \int_{t_0}^{\tau} X^{-1}(t)\, \mathbf{q}(t)\, dt : \mathbf{q}(t) \text{ measurable}, \mathbf{q}(t) \in \boldsymbol{\varphi}(U, t) \text{ for } t_0 \le t \le \tau \right\}.$$

Clearly, $\Lambda_\tau \subset R_\tau$. We shall show that R_τ is compact and convex, and that $R_\tau = \Lambda_\tau$.

Since U is compact, and $\boldsymbol{\varphi}$ and X^{-1} are continuous functions, it follows that R_τ is bounded.

The proof that R_τ is closed and convex is almost identical to the proof of a similar theorem of Blackwell [12], and we shall simply indicate what modifications need be made in the proofs in [12]. The convexity follows directly from a theorem of Liapounoff as in Theorem 3 of [12]. To show that R_τ is closed we need only prove a theorem analogous to Theorem 4 of [12] (taking note of the Lemma on p. 395). Thus, we must show that if $\eta_1, \cdots, \eta_{n+1}$ are $n + 1$ linearly independent vectors in E_{n+1}, and $\lambda_1, \cdots, \lambda_{n+1}$ are numbers defined inductively by

$$\lambda_1 = \min_{\mathbf{r} \in \bar{R}_\tau} \eta_1 \cdot \mathbf{r}, \qquad S_1 = \{\mathbf{r} : \mathbf{r} \in \bar{R}_\tau \text{ and } \eta_1 \cdot \mathbf{r} = \lambda_1\}$$

$$\lambda_i = \min_{\mathbf{r} \in S_{i-1}} \eta_i \cdot \mathbf{r}, \qquad S_i = \{\mathbf{r} : \mathbf{r} \in S_{i-1} \text{ and } \eta_i \cdot \mathbf{r} = \lambda_i\},$$

then the point $\bar{\mathbf{r}} \in E_{n+1}$ which satisfies the relations $\eta_i \cdot \bar{\mathbf{r}} = \lambda_i$, $i = 1, \cdots, n+1$, also belongs to R_τ.

It is clear that $\bar{\mathbf{r}} \in \bar{R}_\tau$ the closure of R_τ. Let $\{\mathbf{r}_j\}$, $\mathbf{r}_j = \int X^{-1}\mathbf{q}_j dt$, be a sequence of points in R_τ such that $\mathbf{r}_j \to \bar{\mathbf{r}}$ as $j \to \infty$. Then, almost precisely as in [12], we can show that there is a subsequence $\{\mathbf{q}_{j_k}\}$ of $\{\mathbf{q}_j\}$, and summable scalar-valued functions $\sigma_i(t)$ for $i = 1, \cdots, n+1$, such that

$$\eta_i \cdot X^{-1}(t)\, \mathbf{q}_{j_k}(t) \xrightarrow[k\to\infty]{} \sigma_i(t) \text{ a.e. in } [t_0, \tau], i = 1, \cdots, n+1; \qquad (6)$$

and

$$\eta_i \cdot \mathbf{r}_{j_k} = \int_{t_0}^{\tau} \eta_i \cdot X^{-1}(t)\, \mathbf{q}_{j_k}(t)\, dt \xrightarrow[k\to\infty]{} \int_{t_0}^{\tau} \sigma_i(t)\, dt$$

$$= \eta_i \cdot \bar{\mathbf{r}} = \lambda_i, i = 1, \cdots, n+1.$$

Since the vectors η_i are linearly independent, the vectors $\eta_i \cdot X^{-1}(t)$ for $i = 1, \cdots, n+1$ are linearly independent for every t. Thus, (6) implies that $\mathbf{q}_{j_k}(t) \to \mathbf{q}(t)$ a.e. as $k \to \infty$ for a certain measurable vector function $\mathbf{q}(t)$. Since $\mathbf{q}_{j_k}(t) \in \boldsymbol{\varphi}(U, t)$ for every t, and the sets $\boldsymbol{\varphi}(U, t)$ are closed, we may assume that $\mathbf{q}(t) \in \boldsymbol{\varphi}(U, t)$ for every t. Finally, $\eta_i \cdot \int X^{-1}\mathbf{q} dt = \lambda_i$ for each i, so that

$$\bar{\mathbf{r}} = \int_{t_0}^{\tau} X^{-1}(t)\, \mathbf{q}(t)\, dt \in R_\tau.$$

The fact that $R_\tau = A_\tau$ follows from a lemma of Filippov [7], which states that every measurable function $\mathbf{q}(t)$ which satisfies the condition $\mathbf{q}(t) \in \boldsymbol{\varphi}(U, t)$, for $t_0 \leq t \leq \tau$ can be realized in the form $\mathbf{q}(t) = \boldsymbol{\varphi}(u(t), t)$ where $u(t)$ is admissible for $t_0 \leq t \leq \tau$.

It remains to show that $\mathscr{A}$ is compact. Because of our assumptions on $\boldsymbol{\varphi}$ and U, it follows that the sets $\mathscr{A}_\tau$ for $\tau \in \Gamma$ are uniformly bounded (we recall that Γ is bounded), so that $\mathscr{A}$ is bounded. To show that $\mathscr{A}$ is closed, consider a sequence of points $(t_i, \mathbf{x}^i)$ belonging to $\mathscr{A}$ and converging to $(t^*, \mathbf{x}^*)$. Since Γ is compact, $t^* \in \Gamma$, so that it suffices to show that $\mathbf{x}^* \in \mathscr{A}_{t^*}$. We shall construct a sequence of points $\mathbf{y}^i \in \mathscr{A}_{t^*}$ such that $\| \mathbf{y}^i - \mathbf{x}^i \| \to 0$ as $i \to \infty$, and our conclusion will follow from the fact that $\mathscr{A}_{t^*}$ is closed. Since $\mathbf{x}^i \in \mathscr{A}_{t_i}$, there is an admissible control $u^i(t)$ such that

$$\mathbf{x}^i = X(t_i) \left[\mathbf{x}^0 + \int_{t_0}^{t_i} X^{-1}(t) \boldsymbol{\varphi}(u^i(t), t)\, dt\right], \qquad i = 1, 2, \cdots.$$

If we define $\mathbf{y}^i$ by

$$\mathbf{y}^i = X(t^*) \left[\mathbf{x}^0 + \int_{t_0}^{t^*} X^{-1}(t) \boldsymbol{\varphi}(v^i(t), t)\, dt\right],$$

where $v^i(t) \equiv u^i(t)$ for $t_0 \leq t \leq t_i$, and $v^i(t) \equiv \nu$ for $t_i < t \leq t^*$ [unless $t_i \geq t^*$, in which case the first relation fully defines $v^i(t)$] where ν is an arbitrary point of U, it is easily seen that the sequence $\{\mathbf{y}^i\}$ has the desired properties. This completes the proof of the theorem.

Theorem 1 has a number of interesting consequences. We discuss the implications on the existence of optimal controls in the next section. The convexity of the sets $\mathscr{A}_\tau$ is useful if it is desired to compute an optimal control by means of the Pontryagin maximum principle [14]. Namely, the normal to an appropriate support hyperplane of one of the $\mathscr{A}_\tau$ determines the initial conditions on the adjoint system whose solution determines the optimal control [15].

Another consequence of Theorem 1 is a generalization of the "bang-bang" principle of LaSalle [4, 5]. Let $\boldsymbol{\psi}(t)$ denote the (closed) convex hull of $\boldsymbol{\varphi}(U, t)$. Suppose that V is a closed subset of U such that $\boldsymbol{\psi}(t)$ is also the convex hull of $\boldsymbol{\varphi}(V, t)$ for every $t \in \Gamma$. Then, let $\mathscr{A}$ be the attainable set defined as before, and let $\mathscr{A}'$ be the restricted attainable set defined in the same way as $\mathscr{A}$, but with U replaced by V. Let us say that a point $(\bar{t}, \mathbf{x}^1) \in E_{n+2}$ is a relaxed attainable point if there is an absolutely continuous trajectory $\mathbf{x}(t)$ in E_{n+2} satisfying the conditions $\mathbf{x}(t_0) = \mathbf{x}^0$ and $\mathbf{x}(\bar{t}) = \mathbf{x}^1$, and the relation

$$\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t) \in \boldsymbol{\psi}(t)$$

almost everywhere in $[t_0, \bar{t}]$. Let

$$\mathscr{A}'' = \{(t, \mathbf{x}) : t \in \Gamma, (t, \mathbf{x}) \text{ is a relaxed attainable point}\}.$$

It is clear that $\mathscr{A}' \subset \mathscr{A} \subset \mathscr{A}''$. By a theorem of Warga [10, Theorem 2.2], $\mathscr{A}'' \subset \bar{\mathscr{A}}'$. By Theorem 1 above, $\mathscr{A}$ and $\mathscr{A}'$ are closed. Hence, $\mathscr{A} = \mathscr{A}' = \mathscr{A}''$. Thus, admissible controls with values restricted to V can "do anything" that admissible controls with values in U can do. The practical importance of this statement to a design engineer is clear. This result in the special case of $\varphi(u, t) = B(t)\, u$ [$B(t)$ being an $n \times r$ matrix], $\alpha \equiv 0$ and $\varphi_0 \equiv 1$, U the unit cube $| u_i | \leq 1$ $(i = 1, \cdots, r)$, and V the vertices of U is LaSalle's "bang-bang" principle.

III. Existence Theorems

We return to the existence problem described in the introduction. Suppose that the closed sets T_t are upper semicontinuous with respect to inclusion; i.e., for every $\epsilon > 0$ and $t \in \Gamma$ there is a $\delta(\epsilon, t)$ such that $T_{t'}$ is contained in an ϵ-neighborhood of T_t whenever $| t' - t | < \delta$. This implies that the set

$$T = \{(t, x) : t \in \Gamma, x \in T_t\}$$

is closed in E_{n+1}, and that the set

$$\hat{T} = \{(t, \xi, x) : (t, x) \in T, \xi \in E_1\}$$

is closed in E_{n+2}. Thus, under the assumptions made in Sections I and II, $\mathscr{A} \cap \hat{T}$ is compact and nonempty, and there is a point $(\bar{t}, \bar{\xi}, \bar{x}) \in \mathscr{A} \cap \hat{T}$ whose coordinate $\bar{\xi}$ is minimal. In other words, there is an optimal control which transfers x^0 to $\mathscr{T}$ (specifically, to the point $\bar{x} \in \mathscr{T}$ in the time $\bar{t}$) with minimum cost $\bar{\xi}$.

In the majority of applications, the sets T_t vary continuously with t, or are even independent of t, so that the semicontinuity property is satisfied. An important particular case is when each T_t consists of a single point $x(t)$.

A variant of the problem described in the introduction is the following. Suppose that the admissible controls are further constrained by requiring that

$$\int_{t_0}^{\bar{t}} \chi_j(u(t), t)\, dt \in M_j \qquad \text{for} \qquad j = 1, \cdots, k, \tag{7}$$

where the χ_j are continuous functions from E_{r+1} to E_1, the M_j are closed subsets of E_1, and $[t_0, \bar{t}]$ is the interval on which the control $u(t)$ is defined. In all other respects, the basic problem is unchanged.

This problem can be reduced to the original form in the following manner. Introduce the k additional equations:

$$\dot{x}_{n+j}(t) = \chi_j(u(t), t), \qquad j = 1, \cdots, k. \tag{8}$$

Denote by $\hat{x}$ the vector $(x_1, \cdots, x_n, x_{n+1}, \cdots, x_{n+k})$, and by $\hat{f}$ the vector function $(f, \chi_1, \cdots, \chi_k)$, so that Eqs. (1) and (8) become

$$\frac{d\hat{x}}{dt} = \hat{f}(\hat{x}(t), u(t), t). \tag{9}$$

Let $\hat{T}_t = T_t \times M_1 \times \cdots \times M_k$, and let $\hat{\mathscr{T}} = \{\hat{T}_t : t \in \Gamma\}$. Note that the $\hat{T}_t$ are closed. The modified problem is now equivalent to fiinding an admissible control $u(t)$ which transfers [through Eq. (9)] the point $\hat{x}^0 = (x^0, 0, \cdots, 0)$ to $\hat{\mathscr{T}}$ with minimal cost. The constraint (7) is no longer explicit so that our problem has been reduced to the original form.

It is clear that constraints of a more general form than (7) can be handled by this technique, but this type is of the most interest in applications.

If Eqs. (1) and (2) have the form (4), Eqs. (9) and (2) have the same form. If the sets T_t are u.s.c., the sets $\hat{T}_t$ are also u.s.c. Thus, under the same assumptions as were made before, an optimal control exists for the generalized problem as well. This result was previously obtained for a particular case [16].

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